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Conferencia plenaria

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Non-additive probabilities and the laws of large numbers

The Law of Large Numbers is a cornerstone of Probability Theory and Statistics, establishing that the arithmetic mean of a sample is close to the population mean whenever the sample is large enough, even if the sample may still be very small with respect to the whole population. Mathematically, the population is allowed to be *infinite*: let $\{\xi_n\}_n$ be a sequence of independent and identically distributed random variables on a probability space $(\Omega, \mathcal{A}, P)$, then

$$\frac{1}{n}\sum_{i=1}^{n}\xi_i \to E[\xi]$$

as soon as the expectation (i.e. integral) in the right-hand side exists. If the convergence is in probability (i.e. in measure), the LLN is called *Weak*, while almost sure (i.e. almost everywhere) convergence yields a *Strong* LLN.

The LLN has also a strong foundational role in that it implies

$$\frac{\text{number of occurences of } A}{\text{number of trials}} \to P(A).$$

From Bernoulli to Borel, some big names in the history of the LLN did not believe that probabilities should necessarily satisfy the additivity property $A \cap B = \emptyset \implies P(A \cup B) = P(A) + P(B)$. Jakob Bernoulli espoused these ideas in the same book (*Ars Conjectandi*, 1713) where he presented the embryo of the LLN, which he called the 'Golden Theorem'. In an ironic twist, Kolmogorov proved his modern version of the LLN in his *Foundations of the Theory of Probability* (1933), whose enormous influence effaced the notion of non-additivity from mainstream probability.

In the subsequent decades, a growing number of problems slowly emerged for which non-additive probabilities or non-additive set functions have been regarded as adequate tools. The renewed interest in this area has never reached a critical mass within Probability Theory, but it can be estimated that several hundred papers are being published per year across a number of math-based scientific communities. Shige Peng's ICM 2010 plenary lecture marks a recent visibility peak of this otherwise underground trend.

The talk will motivate and introduce recent work on the law of large numbers without additivity. Probabilities come in pairs (to each $P(A)$, there corresponds a $\overline{P}(A) = 1 - P(\overline{A})$) and so expectations come in pairs as well. The Strong LLN adopts the form

$$P(\text{expectation} \le \liminf_n \frac{1}{n}\sum_{i=1}^{n}\xi_i \le \limsup_n \frac{1}{n}\sum_{i=1}^{n}\xi_i \le \overline{\text{expectation}}) = 1$$

and the Weak LLN is

$$P(\text{expectation} - \varepsilon < \frac{1}{n}\sum_{i=1}^{n}\xi_i < \overline{\text{expectation}} + \varepsilon) \to 1.$$

However closely this seems to parallel the additive setting, in fact things become strange, wild and fascinating as some solid intuitions from Additive Probability Theory fail. For instance,

1. The 'Weak' LLN can be stronger than the 'Strong' LLN (look at them– how can that be?)

2. A random variable can be ‘independent’ from itself.
3. Sample means may fail to converge almost surely, or they may converge to a non-constant limit, or the distribution of the limit may not coincide with the limit of their distributions.

We will also show how those examples undermine the hasty interpretations of non-additive probabilities and LLNs made by some authors. The moral of the story, and good advice for young mathematicians too, is: do not extend your familiar intuitions to a generalized object without checking them against examples!